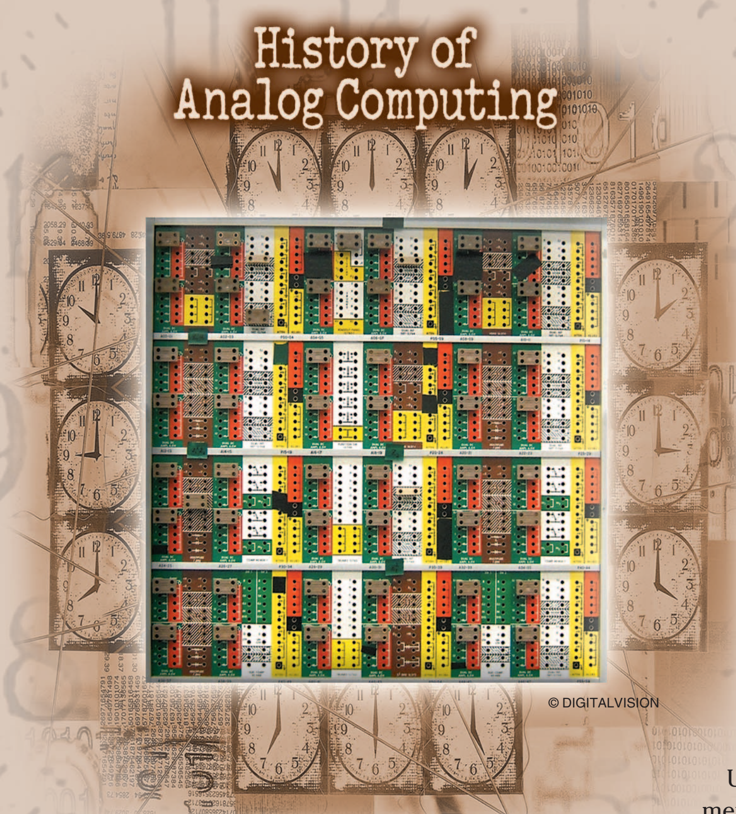
FEATURE

Fundamentals of the Analog Computer

Circuits, technology, and simulation

By Robert M. Howe



The analog computers created in the years immediately following World War II were based on electronic versions of the mechanical differential analyzer, first conceived by Lord Kelvin and implemented at MIT during the 1930s by Vannevar Bush. The heart of the analog computer is the operational amplifier, which consists of a high-gain dc amplifier with a feedback impedance Z_f and input impedances Z_1, Z_2, and Z_3, as shown in Figure 1. Operational amplifiers, configured as in Figure 1, can be combined to solve linear differential equations with constant coefficients. Figure 2 shows how two integrating amplifiers and one summing amplifier can be interconnected to solve a second-order mass-spring-damper system.

The dc operational amplifier was first developed by Philbrick and researchers at Bell Laboratories, but later improved by Goldberg at RCA Laboratories with the addition of drift stabilization. U.S. government-sponsored projects for the development of analog computers for real-time simulation of flight equations included Project Cyclone at Reeves Instrument Corporation, Project Typhoon at RCA Laboratories, and the DACL (Dynamic Analysis and Control Laboratory) at MIT, while Project Whirlwind funded the initial development of digital computers for real-time flight simulation at MIT.

To solve differential equations with specified initial values for the state variables, it is necessary to include initial-condition relays with the integrating amplifiers. Figure 3 illustrates the circuit for a single integrator, which includes both a reset relay and a hold relay. When the reset relay is energized, the amplifier input is switched from the input computing resistors to the summing junction of the two initial-condition resistors; the integrator output voltage then assumes the value $e_o(0)$, the desired initial condition. When the reset relay is de-energized, the

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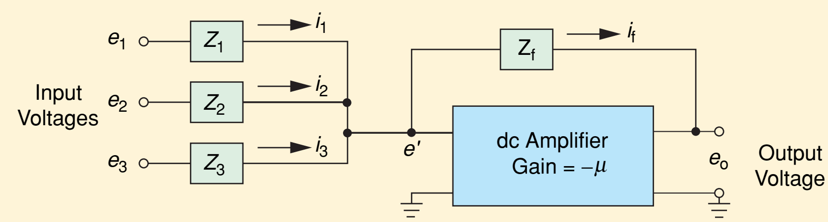
amplifier input is reconnected to the input computing resistors, and the analog solution to the differential equation proceeds. Figure 3 also shows a hold relay that, when energized, disconnects the amplifier input from all input resistors. This procedure then stops the integration and freezes the integrator output voltage at its current value. The hold mode offers a useful method for accurately reading all integrator values at predetermined times during an analog solution.



Letting $i_1+i_2+i_3 = i_f$ and $e' = -\frac{1}{\mu} e_o$, it follows from Ohm's Law that

$$e_o = \frac{\frac{Z_f}{Z_1}e_1 + \frac{Z_f}{Z_2}e_2 + \frac{Z_f}{Z_3}e_3}{1+\frac{1}{\mu}\left[1+\frac{Z_f}{Z_1}+\frac{Z_f}{Z_2}+\frac{Z_f}{Z_3}\right]}.$$

For the dc amplifier gain $\mu >> 1 + \frac{Z_f}{Z_1} + \frac{Z_f}{Z_2} + \frac{Z_f}{Z_3}$,

$$e_o \cong -\left[\frac{Z_f}{Z_1}e_1 + \frac{Z_f}{Z_2}e_2 + \frac{Z_f}{Z_3}e_3\right].$$

Figure 1. *Operational amplifier. When the feedback and input impedances are resistors, the output voltage is proportional to the sum of the input voltages. When a capacitor C is used as a feedback impedance, $Z_f = 1/(Cs)$ and the output voltage is proportional to the time integral of the sum of the input voltages.*

To solve linear differential equations with time-varying coefficients, the author utilized a stepping relay to vary the appropriate computing resistor of an operational amplifier at a fixed sample rate in time. The resistor for each time step was chosen so that the time integral of the resistor value matched the time integral of the corresponding continuous time function. Figure 4 shows a simple example, an analog circuit for solving Bessel's equation of order zero.

In [1], examples are given of analog solutions of the time-dependent transient responses of one- and two-degree-of-freedom mechanical systems. The transient response of simple feedback control systems, where the analog computer solutions are recorded as a function of the independent variable time by using a direct-inking oscillograph, is explored as well. Also shown are the solutions of both Bessel's and Legendre's differential equations, which are solved as second-order linear systems with time-varying coefficients using the stepping-relay scheme described above (and shown in Figure 4) to approximate the variable coefficients. Solutions for the static displacement of structural beams with various boundary conditions and load distributions are also solved by allowing time on the analog computer to correspond to distance along the beam. The normal-mode frequencies of both uniform and nonuniform beams are solved as two-point boundary-value problems, again by letting time on the analog computer represent distance along the beam. Here the unknown ratio of the two nonzero

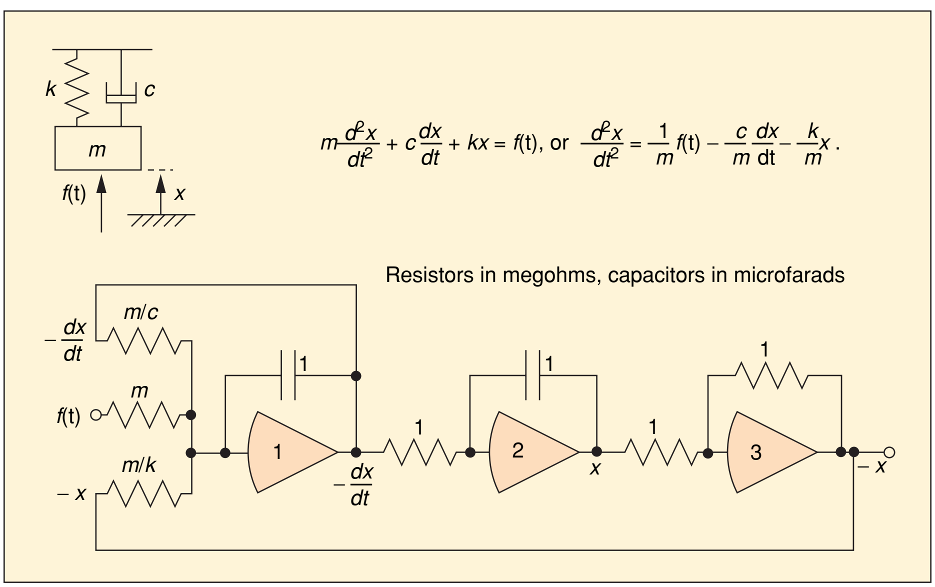


Figure 2. *Analog circuit for simulating a mass-spring-damper system. Amplifier 1 sums the terms in the equation for d^2x/dt^2 and integrates the sum to obtain $-dx/dt$. Amplifier 2 integrates $-dx/dt$ to obtain x. Amplifier 3 simply inverts x to obtain $-x$. Note that all integrators are inverting.*

initial state variables, together with the unknown eigenvalue for each mode, are varied by trial and error until the two specified end conditions are met.

Stability of Operational Amplifiers

Because the operational amplifier represents a feedback system with a wide choice of feedback and input impedances, it is important to consider the dynamic requirements on the amplifier gain $\mu(s)$ to ensure closed-loop amplifier stability. These requirements can be understood by breaking the feedback loop, as shown in Figure 5, with the input impedance Z_i grounded to represent zero input voltages. The overall open-loop transfer function is then given by

$$\frac{e_o}{e'}(s) = -\frac{\mu Z_i}{Z_i + Z_f},$$

also known as the return ratio. For the closed-loop system to be stable, the phase shift of the open-loop system must be more positive than $-\pi$ at the frequency of unity open-loop gain, defined as the crossover frequency. To be conservative, we usually require the phase shift of the open-loop system to be equal to $-\pi/2$ at the crossover frequency. Clearly

$$|\mu| = \left|\frac{Z_f + Z_i}{Z_i}\right|$$

at the crossover frequency. If the feedback and input impedances are resistors, as in the case of a summing amplifier, then the crossover frequency occurs when

$$|\mu| = \frac{R_f + R_i}{R_i}.$$

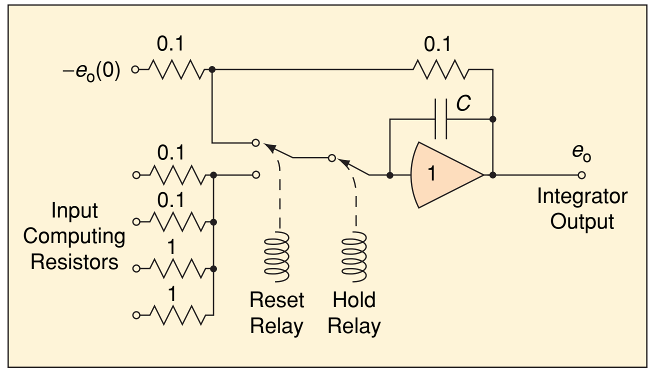


Figure 3. *Circuit for integrator mode control. The reset relay, when energized, establishes the initial condition for the integrator. The hold relay, when energized, stops the integration and freezes the output voltage e_o.*

If the feedback impedance is a capacitor, then $|Z_f| \ll |Z_i|$ and the crossover frequency occurs when $|\mu| = 1$. To ensure that the open-loop phase shift is equal to $-\pi/2$ at the crossover frequency for all possible summer and integrator configurations, it follows that the phase shift of the amplifier gain $\mu(s)$ should be $-\pi/2$ or greater at all frequencies for which $|\mu| > 1$. When the dc operational amplifier is a stable minimum-phase system (no zeros or poles in the right-half plane), which is indeed the case for both vacuum-tube and transistor amplifiers, a phase shift of $-\pi/2$ or greater is realized when the slope of the $\log|\mu|$ versus $\log|\omega|$ plot is greater than or equal to -6 dB/octave. Figure 5 shows the resulting open-loop frequency response.

An example of a more complex configuration of feedback and input impedances is the single amplifier circuit for simulating a second-order linear system, as shown in Figure 6. To examine the stability of this feedback system, we ground the input terminal $-e_i$, break the feedback loop as shown in the figure, and let the open-loop input be e_o'. This one-amplifier circuit for simulating a second-order system can be used in the feedback loop of a high-gain amplifier to

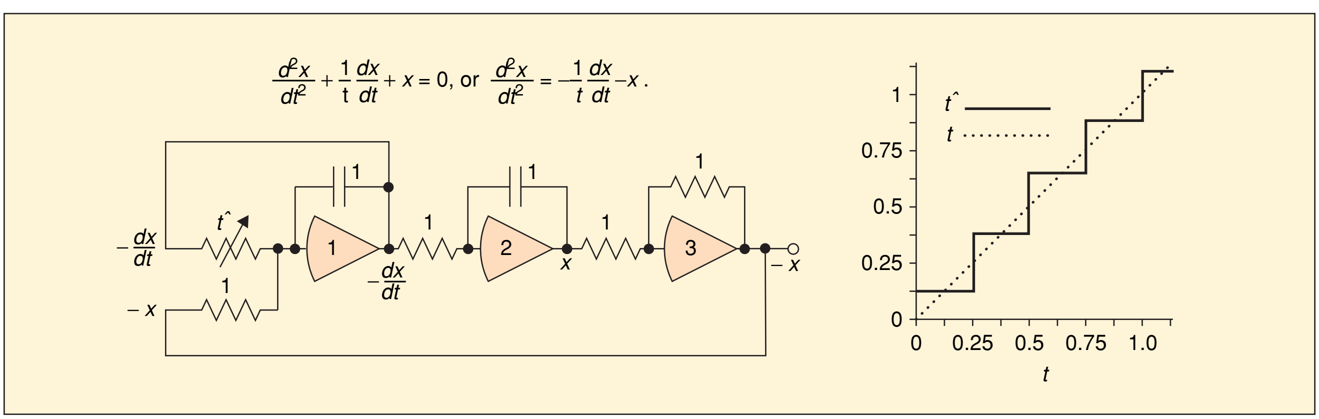


Figure 4. *Analog circuit for solving Bessel's equation of order zero. The variable coefficient is approximated with the staircase function using a stepping relay.*

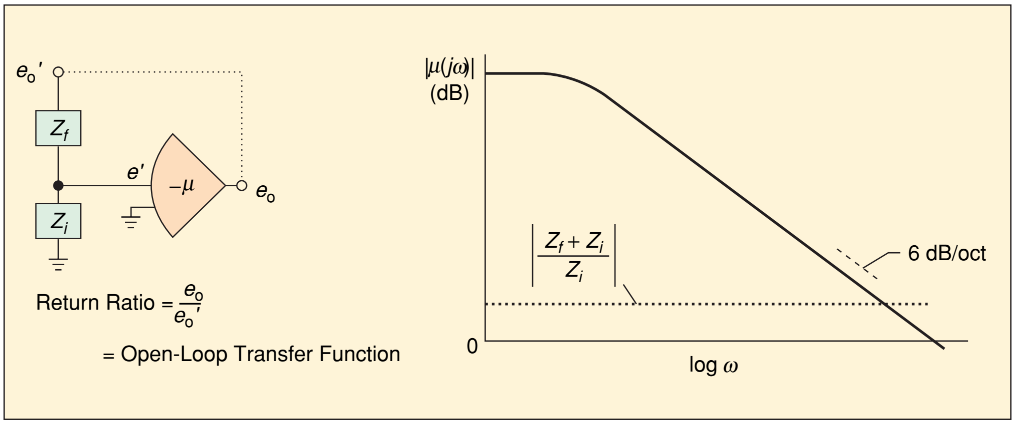


Figure 5. *Operational amplifier frequency response. This open-loop frequency response is needed to provide 90° of phase margin for any ratio of feedback to input impedances.*

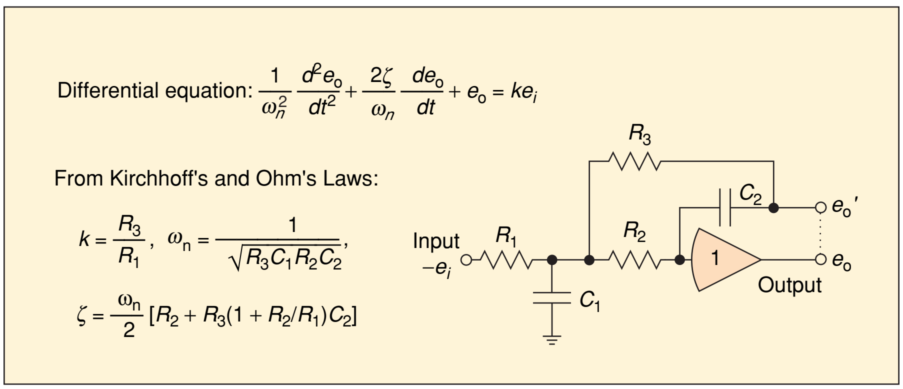


Figure 6. *Single-amplifier circuit for simulating a second-order system when $e_o' = e_o$. For the closed-loop system to be stable with a phase margin of $\pi/2$, the open-loop frequency response should exhibit a slope of -6 dB/octave at and around the frequency of unity (0 dB) open-loop gain.*

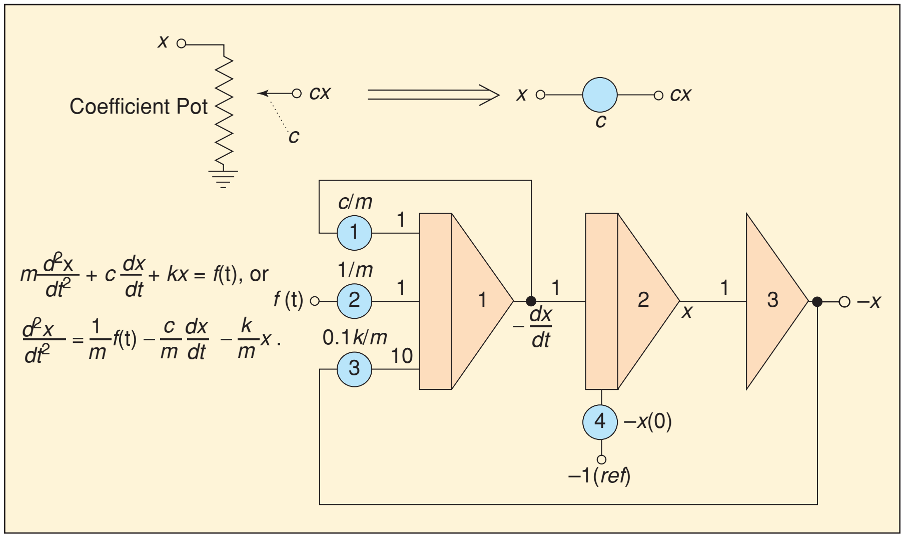
obtain a representation of the reciprocal of a second-order system. The author utilized such a scheme to implement an electronic drive circuit for the two-channel direct-inking oscillograph used for recording analog solutions. In this case, the recorder by itself was well represented by the dynamics of a second-order system with an undamped natural frequency of approximately 30 Hz. When combined with the electronic drive circuitry described earlier, the bandwidth of the overall recording system was extended to more than 100 Hz.

The Use of Potentiometers to Set Coefficient Values in Analog Simulations

All of the circuits described thus far for the analog solution of differential equations have utilized resistors to set summing and integrating amplifier gains. However, a much more practical method for implementing the precision setting of arbitrary coefficients in analog simulations utilizes potentiometers. For example, the potentiometer shown at the top of Figure 7 with input voltage x provides an output voltage cx, where c is a coefficient that can be set between 0 and 1. The diagram of the analog circuit for simulating the mass-spring-damper system, shown earlier in Figure 2, is repeated in Figure 7 using potentiometers to represent the fixed coefficients in the problem. The feedback impedances are not shown in the operational amplifiers in Figure 7. Instead, only the fixed input gains (nominally one or ten) are shown, with a triangle representing a summing amplifier and a triangle attached to a vertical bar representing an integrating amplifier. Thus, in Figure 7, amplifiers 1 and 2 are integrators, whereas amplifier 3 is an inverter. The input resistors m/c and m in Figure 2 are replaced in Figure 7 with coefficient pots 1 and 2 set at c/m and $1/m$, respectively, in both cases driving unity gain inputs. The input resistor m/k in Figure 2 is replaced by coefficient pot 4, set at 0.1 c/m and driving a gain-of-ten input. Pot 4 is used in Figure 7 to set the initial condition $x(0)$ on the mass displacement, whereas there is no indication of initial-condition circuitry in Figure 2.

It should be noted that the setting of the output wiper arm on the coefficient pot must take into account the input resistor driven by the pot, since that input resistor is in parallel with the pot resistance from wiper arm to ground. In general-purpose analog computers, the coefficient pots are normally set in a pot-set mode, with the pot input x replaced by +1 reference (that is, +100 V for analog computers based on 100-V reference, and +10 V for analog computers based on 10-V reference). The pot is then set to the desired value, either by reading a digital voltmeter or by means of a null meter connected between the pot output and a reference pot with a precision dial. Note that the coefficient pots are not set by reading the pot dials, even though the coefficient pots are usually ten-turn helical pots with precision, counter-type dials.

It is apparent that the variables used in analog simulations must be properly scaled both to avoid over-ranging amplifier outputs and to ensure that amplifier outputs range over a reasonable fraction of full-scale during any given solution. A similar problem occurs when using fixed-point digital computation. The simplest solution is to represent each variable as a scaled fraction, the variable divided by its maximum value. Then each scaled-fraction variable will range over ± 1, which corresponds to ± 100 V in the case of 100-V analog computers and ± 10 V in the case of 10-V analog computers. Actually, many engineers (including the author) have found that the process of scaling problem variables can lend considerable insight into the problem itself, including the identification of dominant terms.



$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t), \text{ or}$$

$$\frac{d^2x}{dt^2} = \frac{1}{m}f(t) - \frac{c}{m}\frac{dx}{dt} - \frac{k}{m}x.$$

$$-\frac{dx}{dt}$$

Figure 7. *Using potentiometers to set coefficient values in an analog simulation circuit. Note that pot 3 is set at 0.1 c/m because of the gain-of-10 input. To speed up the setting of coefficients in large problems, servo-set potentiometers were widely used in state-of-the-art analog computers.*

Servomultipliers for Analog Computers

All of the example analog simulations described thus far have involved the solution of linear differential equations, including differential equations with time-varying coefficients. Figure 8 shows a schematic of a typical servomultiplier. The three ganged potentiometers at the right side of the figure are driven by a two-phase induction motor with an n:1 gear reduction (not shown in the figure). The servomotor is in turn driven by a magnetic amplifier, which receives its dc input from an analog controller unit. The error in servo output angle is given by $e = X_{\text{in}} - X_{\text{out}}$, where X_{in} is the command input angle for the servo and X_{out} is the measured output angle, as obtained from the wiper arm output $-X_{\text{out}}$ of the potentiometer with -100 V and $+100$ V connected to the high and low side, respectively. With $\pm Y$ and $\pm Z$ connected across the second and third potentiometers, the pot outputs represent XY and XZ, respectively. Note that the static accuracy of the servomultiplier depends on the linearity of each ganged pot as well as the static nulling error of the servo.

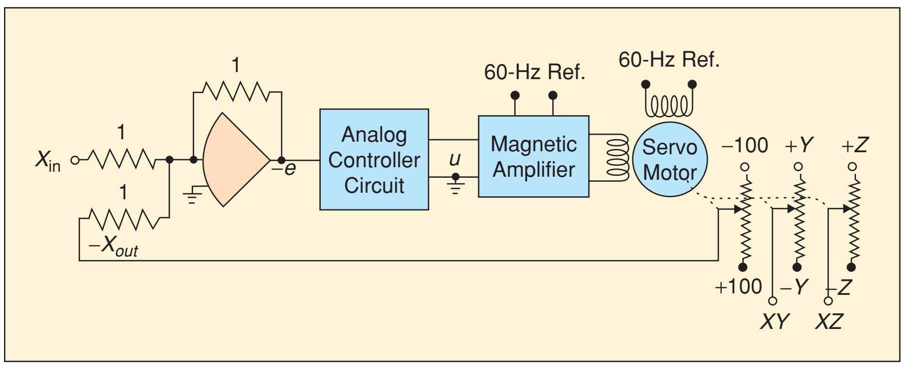


Figure 8. *A servomultiplier that uses a 60-Hz, two-phase servo motor driven by a magnetic amplifier. The analog controller circuit is used to provide additional damping. Typical multiplier accuracy was 0.1% or better. Because of the limited servo bandwidth, the servo input X_{in} should be assigned to the slower of the two variables in computing a product.*

Typical pots used in servomultipliers exhibit a maximum linearity error ranging between 0.02% for ten-turn wire-wound, ganged potentiometers to 0.1% for one-turn ganged potentiometers, either wire wound or film type. The static nulling error of the servomultiplier depends on the overall design of the servo, the static friction due to both pot friction and gear-train friction, and the static resolution of the ganged potentiometers. In the case of wire-wound potentiometers, the static resolution is equal to the reciprocal of the number of wire turns, typically 3,000 for one-turn pots and more than 10,000 for ten-turn pots. The static resolution error associated with conducting-film pots can approach zero. To preserve static accuracy in the servomultiplier, it should also be noted that it is important for the multiplier pots with outputs XY and XZ in Figure 8 to drive input resistors that are identical with the resistor that loads the output $-X_{\text{out}}$ of the reference pot (that is, 1 megohm in the example shown here).

The transfer function of a typical servomotor plus the magnetic amplifier driver is given by

$$M(s) = \frac{K_m/I}{s^2 + s/T_m}, \qquad (1)$$

where I is the total inertia of the servomultiplier (including servomotor, gear train, and ganged potentiometers referred to the servo output shaft) and T_m is the motor time constant. If the analog controller circuit in Figure 8 consists of a pure gain K, then the overall servo transfer function is approximately represented by a second-order system with an undamped natural frequency given by $\omega_n = \sqrt{KK_m/I}$ and damping ratio given by $\zeta = (1/2)\sqrt{I/KK_m}/T_m$. To obtain satisfactory nulling errors, the gain constant K for typical servomultipliers must be sufficiently large that the resulting damping ratio ζ is too small for satisfactory transient performance. Thus, it is necessary to add rate control to the proportional control in the analog controller circuit. It also may be desirable to add integral control to ensure low nulling error (that is, good servo tracking) for a slowly varying input X_{in}. Figure 9 shows an operational amplifier circuit that provides the necessary controller transfer function, where the rate control term $C_e s$ is bandwidth-limited by the transfer function $1/(\tau s + 1)$.

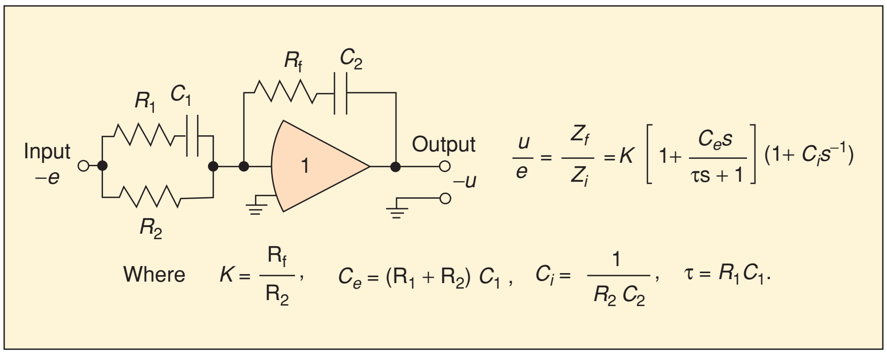


Figure 9. *One-amplifier analog controller circuit for proportional, bandwidth-limited rate and integral control. Alternatively, a circuit that requires two conventional integrating amplifiers and two summing amplifiers can be used.*

The overall servomultiplier transfer function for sinusoidal inputs predicts the dynamic performance of the closed-loop system based on linear models of the subsystems. However, the servomotor has two important characteristics that are not modeled by the motor transfer function (1). These characteristics include the acceleration limit $|X''|_{\max}$ and the velocity limit $|X'|_{\max}$ for large motor control inputs. The experience gained in designing servomultipliers for analog computers led to the realization that the design of an electromechanical control system, including the choice of components, is dominated by nonlinear requirements such as the acceleration and velocity limits (large-motion nonlinearities) and the nulling error or static resolution (small-motion nonlinearities). The role of linear design methods is generally limited to the choice of controller parameters needed to produce satisfactory system stability margin and transient response [2].

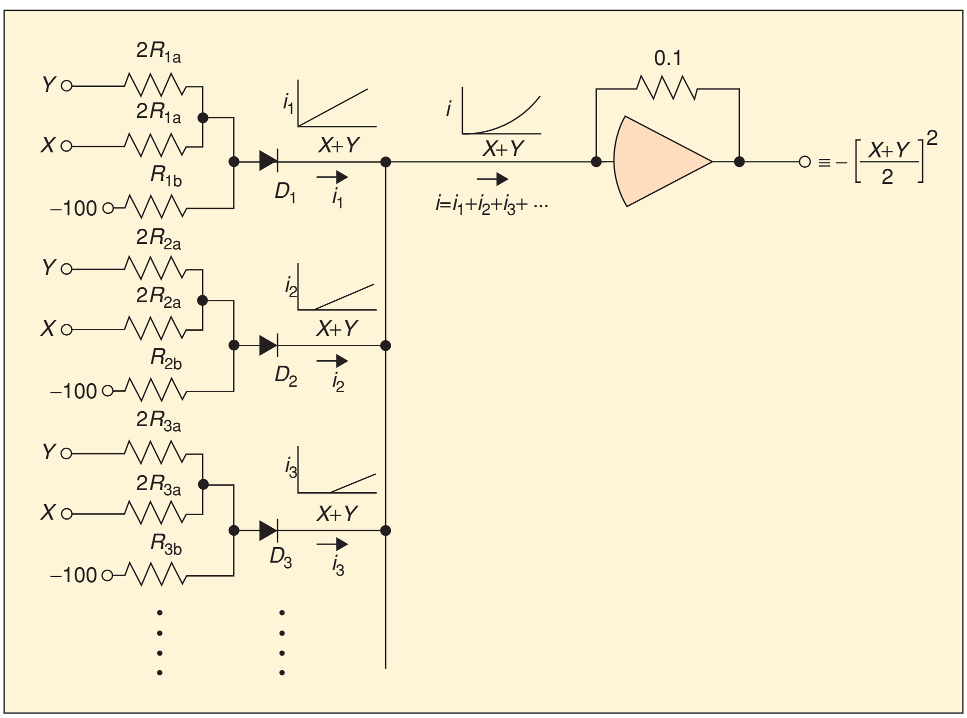


Figure 10. *Diode circuit for approximating $((X+Y)/2)^2$ for $X+Y > 0$. The same circuit is repeated with inputs $-X$ and $-Y$ to approximate $((X+Y)/2)^2$ for $X+Y < 0$. A similar pair of circuits with inputs X and $-Y$, and $-X$ and Y, with diodes reversed and $+100$ bias voltages, is used to approximate $-((X-Y)/2)^2$. A single operational amplifier is used to sum the outputs of all four circuits to produce the product XY.*

Other Nonlinear Analog-Computer Components

While the servomultipliers described above were widely utilized in analog computers during the 1950s, several alternative, all-electronic devices were developed to improve the overall dynamics of analog multipliers. The first of these was the time-division multiplier, initially developed by Goldberg [3]. In this scheme, the voltage X is converted to a pulse-width modulated signal used to drive an electronic switch, which modulates the voltage Y. The area under each cycle of the switched signal is proportional to the product XY. The switched signal drives a low-pass filter to produce a smoothed output XY. Although time-division multipliers achieved dynamic errors that were typically two orders of magnitude smaller than those associated with servomultipliers, these multipliers often exhibited undesirable drift in voltage offset with time. Another device was the quarter-square multiplier, which was based on the equation

$$XY = \frac{(X+Y)^2 - (X-Y)^2}{4}. \tag{2}$$

The circuit shown in Figure 10 is used to generate a segmented approximation to $((X+Y)/2)^2$ for $X+Y>0$. When $X+Y$ becomes positive enough, the diode D_1 starts conducting and the current i_1 starts to increase linearly for further increases in $X+Y$. For a larger value of $X+Y$, diode D_2 starts to conduct, and the current i_2 starts to increase linearly with increasing $X+Y$. The overall circuit in Figure 10 consists of n diode segments. The amplifier output in Figure 10 then produces an n-segment approximation to $-((X+Y)/2)^2$ for $X+Y>0$. With ideal diodes, the segmented approximation to a quadratic function then exhibits a maximum fractional error of $1/(8n^2)$. With the silicon-junction diodes normally used for quarter-square multipliers, the nonideal diode current-voltage characteristic produces additional rounding of each segment knee, further reducing the quadratic-approximation error.

It should be noted that (2) for the quarter-square multiplier can be replaced with the formula

$$XY = \frac{|X+Y|^2 - |X-Y|^2}{4}.$$

By replacing $(X+Y)^2$ with $|X+Y|^2$ and replacing $(X-Y)^2$ with $|X-Y|^2$, we can eliminate the need for two of the four segmented quadratic approximation circuits similar to the one shown in Figure 10. Furthermore, the calculation of $|X+Y|$ and $|X-Y|$ can be combined with the diode circuit for the segmented quadratic approximation by using the circuit shown in Figure 11. Here the voltage at the junction of diodes D_{a1} and D_{a2} is proportional to $|X+Y|/2$, and the voltage at the junction of diodes D_{a3} and D_{a4} is proportional to $-|X-Y|/2$, as indicated in Figure 11. These diode-junction voltages then drive biased-diode circuits similar to the one in Figure 10 to produce segmented approximations to the quadratic function. Because of the loading effect of the biased-diode circuits on the passive absolute-value circuit in Figure 11, the effect of the nonideal diode characteristics on the rounding of the knees of the segmented approximation is increased. This rounding in turn causes error in the segmented approximation to the quadratic function to be considerably less than the value of $1/(8n^2)$ for ideal diodes, where n is the number of diode segments used for the quadratic approximation. Thus, the completely passive quarter-square multiplier circuit of Figure 11, in addition to reducing by

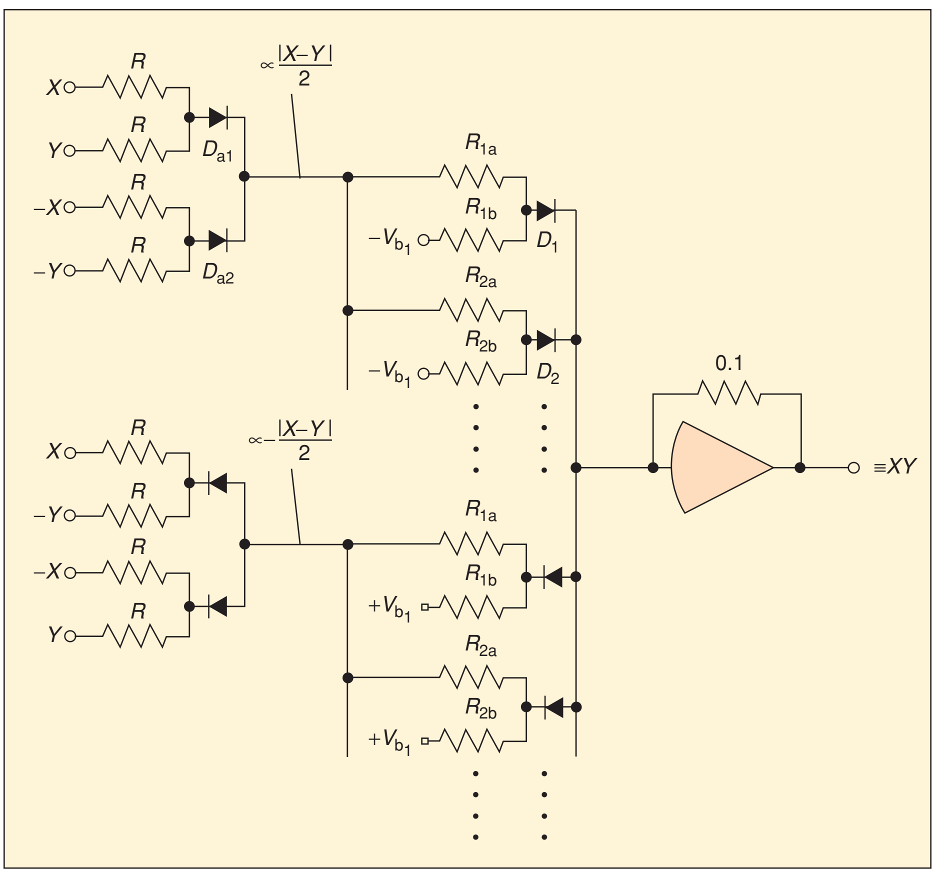


Figure 11. *A completely passive quarter-square multiplier circuit terminated in a single operational amplifier. The circuit is based on $XY = (|X+Y|/2)^2 - (|X-Y|/2)^2$. The circuit output is connected to the summing junction of an operational amplifier with a 0.1 mΩ feedback resistor.*

more than a factor of two the number of required components, also is more accurate than the circuit in Figure 10 for a given number n of diode segments. The author developed a ten-segment version of this circuit in 1959.

It should be noted that analog division of Y by X can be accomplished by utilizing a multiplier in the feedback loop of an operational amplifier. With the amplifier input connected to $-Y$, the multiplier input X_{in} connected to X, the multiplier input Y_{in} connected to the amplifier output, and the multiplier output $X_{\text{in}}Y_{\text{in}}$ connected to the amplifier feedback resistor, the amplifier output is the quotient Y/X.

The generation of a function $f(X)$ can be accomplished in analog computation by using a segmented approximation to $f(X)$. The segmented approximation can be generated using a diode function generator, which consists of biased-diode circuitry similar to that shown in Figure 10, with potentiometers used to set the slope and breakpoint of each diode segment. Alternatively, a servomultiplier can be used, with n taps on one of the servo-driven pots. Padding resistors connected between the taps can then be adjusted to obtain the pot output $Yf(X)$, with $f(X)$ represented by an approximation with $n+1$ straight-line segments. Similarly, the functions $\sin(X)$ and $\cos(X)$ are generated either by diode function generators or servo-driven sine-cosine pots. For a description of these and other analog components, see [4]. One of the first implementations of time-optimal control was accomplished using biased-diode analog circuitry to represent the required nonlinear control law [5]. Also, the output saturation voltage characteristics of unstabilized operational amplifiers were used to represent simple nonlinearities, such as relay-type bang-bang controllers and Coulomb friction [6].

Current Availability of Analog Components

It should be noted that many of the components utilized in the early analog computers are currently available in integrated-circuit form. For example, operational amplifiers with bandwidths and drift characteristics equal to or better than the drift-stabilized amplifiers of two or three decades ago are available at a fraction of the cost, with up to four amplifiers located on a single chip. Nevertheless, with the current and projected speed of digital-processor chips, it seems unlikely that analog computers will ever again match their prominent former role as a general-purpose device for the simulation of dynamic systems.

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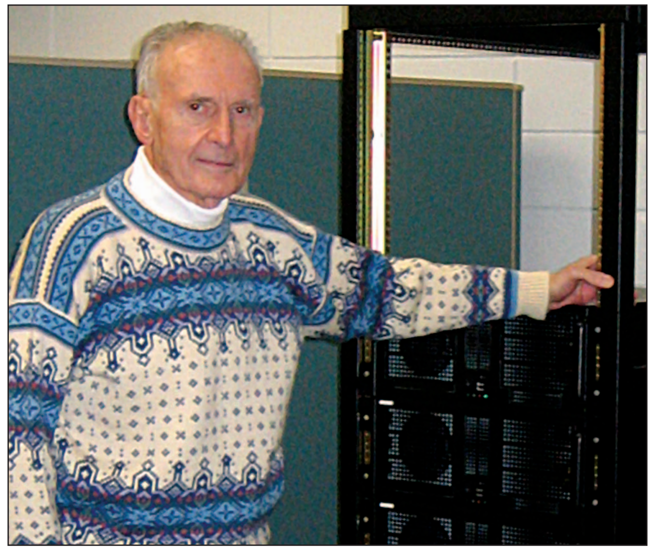

Robert M. Howe, one of the founders of Applied Dynamics International (ADI) and Professor Emeritus at the University of Michigan, where he taught for 41 years. This image shows one of the latest ADI products, a distributed rtX system utilizing multiple PCs for real-time hardware-in-the-loop simulation.

Robert M. Howe (bobhowe@umich.edu) is Professor Emeritus of Aerospace Engineering at the University of Michigan, where he served on the faculty for 41 years, including 15 years as department chair. He received a B.S. in electrical engineering from Caltech. He received an A.B. from Oberlin College, an M.S. from the University of Michigan, and a Ph.D. from MIT, all in physics. His interests include real-time simulation, as well as flight dynamics and control. He is the author of over 100 technical papers and one book on analog computers. He served as the first national chair of Simulation Councils, Inc., the predecessor of the Society for Computer Simulation. His many honors include the 1983 AIAA de Florez Training Award for Flight Simulation and the 1978 Award for Meritorious Civilian Service for "outstanding contributions in guidance and control, and flight simulation" while a member of the USAF Scientific Advisory Board. He has served as a technical consultant to many companies. His technical society memberships include AIAA (Associate Fellow), SCS, and IEEE (Fellow). He was a founder of Applied Dynamics International, an Ann Arbor computer company for which he continues to consult since his retirement in 1991 from the University of Michigan. He can be contacted at 485 Rock Creek Dr., Ann Arbor, MI 48104 USA.